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Quantum line bundles on a noncommutative sphere

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Abstract
A noncommutative (NC) sphere is introduced as a quotient of the enveloping algebra of the Lie algebra $su(2)$. Following Gurevich and Saponov (2001 *J. Phys. A: Math. Gen.* **34** 4533–69) and using the Cayley–Hamilton identities we introduce projective modules which are analogues of line bundles on the usual sphere (we call them quantum line bundles) and define a multiplicative structure in their family. Also, we compute a pairing between quantum line bundles and finite-dimensional representations of the NC sphere in the spirit of the NC index theorem. Moreover, we introduce projective modules being NC counterparts of tangent vector bundles and those of differential forms and define an analogue of the de Rham complex making use of these modules.

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1. Introduction

One of the basic notions of the usual (commutative) geometry is that of the vector bundle on a variety. As was shown in [Se] the category of vector bundles over a regular affine algebraic variety X is equivalent to the category of finitely generated projective modules over the algebra $\mathcal{A} = \mathbb{K}(X)$ which is the coordinate ring of the given variety X (a similar statement for smooth compact varieties was shown in [Sw]). Hereafter $\mathbb{K}$ stands for the basic field (always $\mathbb{C}$ or $\mathbb{R}$).

The language of projective modules is perfectly adapted to the case of a noncommutative (NC) algebra $\mathcal{A}$. Any such (say, right) $\mathcal{A}$-module can be identified with an idempotent $e \in M_n(\mathcal{A})$ for some natural n. These idempotents play a key role in all approaches to NC geometry, in particular, in a NC version of the index formula of Connes ([C, L]).

The problem of constructing projective modules over physically meaningful algebras is of great interest. The $\mathbb{C}^*$-algebras (apart from commutative ones) are mostly studied from this viewpoint. Besides, there are very few examples known. As an example let us evoke the paper [Ri] where projective modules over NC tori are studied (also cf [KS] and the references therein). In recent times a number of papers have appeared dealing with some algebras (less standard than NC tori and those arising from the Moyal product) for which certain projective modules are constructed by hand (cf for example [DL, LM] and the references therein).

Nevertheless, there exists a natural method suggested in [GS] of constructing projective modules over NC analogues of $\mathbb{K}(\mathcal{O})$, where $\mathcal{O}$ is a generic[3] $SU(n)$-orbit in $su(n)^*$ including its 'q-analogues' arising from the so-called reflection equation (RE) algebra. This method is based on the Cayley–Hamilton (CH) identity for matrices with entries belonging to the NC algebras in question. (However, it seems very plausible that other interesting examples of 'NC varieties' can be covered by this method. In particular, by making use of the CH identity for super-matrices, cf [KT], it is possible to generalize our approach to certain super-varieties.)

The idea of the method consists of the following. Consider a matrix $L = \|l_j^i\|$ subject to the RE related to a Hecke symmetry (cf [GPS])[4]. Then it satisfies a polynomial identity

$$L^p + \sum_{i=1}^{p} a_i(L) L^{p-i} = 0 \tag{1.1}$$

where coefficients $a_i(L)$ belong to the centre of the RE algebra generated by l_j^i.

Passing to a specific limit in the RE algebra we get a version of the CH identity for the matrix whose entries l_j^i commute as follows:

$$\left[l_j^i, l_l^k\right] = \hbar\left(\delta_j^k l_l^i - \delta_l^i l_j^k\right) \tag{1.2}$$

i.e. l_j^i generate the Lie algebra $gl(n)_\hbar$ where $\mathbf{g}_\hbar$ stands for a Lie algebra whose Lie bracket equals $\hbar[\ ,\ ]$, where $[\ ,\ ]$ is the bracket of a given Lie algebra $\mathbf{g}$. Introducing the parameter $\hbar$ allows us to consider the enveloping algebras as deformations of commutative ones. (Let us note that this type of CH identity has been known since 1980s, cf [Go].)

Similar to the general case the coefficients of the corresponding CH identity belong to the centre $Z[U(gl(n)_\hbar)]$ of the enveloping algebra $U(gl(n)_\hbar)$. Therefore, by passing to a quotient[5]

$$U(gl(n)_\hbar)/\{z - \chi(z)\}$$

where $z \in Z[U(gl(n)_\hbar)]$ and

$$\chi : Z[U(gl(n)_\hbar)] \to \mathbb{K}$$

is a character we get a CH identity with numerical coefficients

$$L^p + \sum_{i=1}^{p} \alpha_i L^{p-i} = 0 \qquad \alpha_i = \chi(a_i(L)). \tag{1.3}$$

Consider now the corresponding polynomial equation

$$\lambda^p + \sum_{i=1}^{p} \alpha_i \lambda^{p-i} = 0 \tag{1.4}$$

and assume its roots to be pairwise distinct. Then we can assign an idempotent (or what is the same, a one-sided projective module) to each root. For quasiclassical Hecke symmetries (see footnote 4) these projective modules are deformations of line bundles on the corresponding classical variety. For this reason we call them *quantum line bundles* (q.l.b.).

3 An orbit is called generic if it contains a diagonal matrix with pairwise distinct eigenvalues.

4 A Hecke symmetry is a Hecke-type solution of the quantum Yang–Baxter equation. There exist different types of Hecke symmetries: quasiclassical ones being deformations of the classical flip and non-quasiclassical ones (a big family of such Hecke symmetries was introduced in [G]). Let us note that a version of the CH identity exists for any of them independently of the type.

5 Hereafter $\{X\}$ stands for the ideal generated by a set X in the algebra in question.

In this paper, we constrain ourselves to the case arising from the Lie algebra $gl(2)$. Namely, we describe a family of projective modules over the algebra

$$\mathcal{A}_\hbar = U(sl(2)_\hbar)/\{\Delta - \alpha\} \tag{1.5}$$

where Δ stands for the Casimir element in the algebra $U(sl(2)_\hbar)$.[6] We consider this algebra as a NC counterpart of a hyperboloid.

In order to get a NC counterpart of the sphere we should pass to the compact form of the algebra in question. However, it does not affect the CH identity since it is indifferent to a concrete form (compact or not) of the algebra.

To describe our method in more detail we begin with the classical (commutative) case. Put the matrix

$$L = \begin{pmatrix} \mathrm{i}x & -\mathrm{i}y + z \\ -\mathrm{i}y - z & -\mathrm{i}x \end{pmatrix}$$

in correspondence to a point $(x, y, z) \in S^2$. This matrix satisfies the CH identity (1.3) where $p = 2, \alpha_1 = 0, \alpha_2 = x^2 + y^2 + z^2 = \text{ const } \neq 0$.

Let λ_1 and $\lambda_2 = -\lambda_1$ be the roots of equation (1.3). To each point $(x, y, z) \in S^2$ we assign the eigenspace of the above matrix L corresponding to the eigenvalue $\lambda_l, l = 1, 2$. Thus, we come to a line bundle E_l which will be called *basic*.

Various tensor products of the basic bundles $E_l, l = 1, 2$, give rise to a family of *derived or higher* line bundles

$$E^{k_1,k_2} = E_1^{\otimes k_1} \otimes E_2^{\otimes k_2} \qquad k_1, k_2 = 0, 1, \ldots.$$

Note, that certain line bundles of this family are isomorphic to each other. In particular, we have $E_1 \otimes E_2 = E^{0,0}$ where $E^{0,0}$ stands for the trivial line bundle. In general, the line bundles

$$E^{k_1,k_2} \quad \text{and} \quad E^{k_1+l,k_2+l} \qquad l = 1, 2, \ldots$$

are isomorphic to each other. So, any line bundle is isomorphic either to E_1^k or to E_2^k for some $k = 0, 1, \ldots$ (we assume that $E_1^0 = E_2^0 = E^{0,0}$). Finally, we conclude that the Picard group $\mathrm{Pic}(S^2)$ of the sphere (which is the set of classes of isomorphic line bundles equipped with the tensor product) is nothing but $\mathbb{Z}$ since any line bundle can be represented as E_1^k with a proper $k \in \mathbb{Z}$, where we put $E_1^k = E_1^{\otimes k}$ for $k > 0$ and $E_1^k = E_2^{\otimes(-k)}$ for $k < 0$.

It is worth emphasizing that we deal with an algebraic setting: the sphere and total spaces of all bundles in question are treated as real or complex affine algebraic varieties (depending on the basic field $\mathbb{K}$).

Now, let us pass to the NC case. Using the above NC version of the CH identity one can define NC analogues $E_l(\hbar), l = 1, 2$, of the line bundles E_l. We will call them *basic* q.l.b. (they are defined in section 2).

Unfortunately, for a NC algebra $\mathcal{A}$ the tensor product of two or more one-sided (say, right) $\mathcal{A}$-modules is not well defined. So, the construction of derived line bundles cannot be generalized to a NC case in a straightforward way. Nevertheless, using the CH identities for some 'extensions' $L_{(k)}, k = 2, 3 \ldots$ of the matrix L to higher spins (in the following $k = 2 \times$ spin) we directly construct NC counterparts $E^{k_1,k_2}(\hbar)$ of the above derived line bundles. Thus, for any $k = 2, 3, \ldots$ (with $k_1 + k_2 = k$) we have $k + 1$ *derived* q.l.b.

In section 3, we explicitly calculate the CH identity for the matrix $L_{(2)}$ and state that such an identity exists for any matrix $L_{(k)}, k > 2$.

[6] Note that in fact we deal with the Lie algebra $sl(2)_\hbar$ instead of $gl(2)_\hbar$ since the trace of the matrix L is always assumed to vanish. The studies of such quotients were initiated by Dixmier (cf [H] and the references therein). In certain papers the compact form of this algebra is called a *fuzzy* sphere.

However, as usual we are interested in projective modules (in particular, q.l.b.) modulo natural isomorphisms. Thus, we show that the q.l.b. $E^{1,1}(\hbar)$ is isomorphic to the trivial one $E^{0,0}(\hbar)$ which is nothing but the algebra $\mathcal{A}_\hbar$ itself. We also conjecture that the q.l.b.

$$E^{k_1,k_2}(\hbar) \quad \text{and} \quad E^{k_1+l,k_2+l}(\hbar) \tag{1.6}$$

are isomorphic to each other. If so, any q.l.b. is isomorphic to $E^{k,0}(\hbar)$ or $E^{0,k}(\hbar)$ similar to the commutative case.

Then we define an associative product in the family of q.l.b. over the NC sphere in a natural way. By definition, the product of two (or more) q.l.b. over the NC sphere is the NC analogue of the product of their classical counterparts. Otherwise stated, we set by definition

$$E^{k_1,k_2}(\hbar) \cdot E^{l_1,l_2}(\hbar) = E^{k_1+l_1,k_2+l_2}(\hbar)$$

(in particular, we have $E_1(\hbar) \cdot E_2(\hbar) = E^{1,1}(\hbar)$ and in virtue of the above-mentioned result this product is isomorphic to $E^{0,0}(\hbar)$). The family of all modules $E^{k_1,k_2}(\hbar)$ equipped with this product is a semigroup. It is denoted prePic($\mathcal{A}_\hbar$) and called prePicard. Assuming the above conjecture to be true we get the Picard group Pic($\mathcal{A}_\hbar$) of the NC sphere (it is a group since any of its elements becomes invertible)[7]. All these notions are introduced in section 4. Moreover, in this section we compute the pairing between the q.l.b. in question and irreducible representations of the algebra $\mathcal{A}_\hbar$ in the spirit of the NC index theorem.

In the last section we consider NC analogues of some 'geometrical' vector bundles, i.e. those coming in differential calculus. More precisely, we present analogues of tangent vector bundles and those of differential forms. The latter objects are used in order to define a new version of the NC de Rham. We complete the paper by comparing our version of the NC de Rham complex with others known in this area.

To complete the introduction we emphasize that our approach is in principle applicable to the NC analogue of any generic orbit in $su(n)^*$ (and even to their 'q-analogues' arising from the RE algebra) but the calculation of the higher CH identities becomes much more difficult as the degree of the CH identity for the initial matrix becomes greater than 2. We refer the reader to the paper [GLS] where the mentioned 'q-analogues' of the above NC sphere are considered.

2. Basic quantum line bundles on the NC sphere

Consider the Lie algebra $gl(2)_\hbar$ which is generated by the elements a, b, c and d satisfying the following commutation relations:

$$[a,b] = \hbar b \qquad [a,c] = -\hbar c \qquad [a,d] = 0$$
$$[b,c] = \hbar(a-d) \qquad [b,d] = \hbar b \qquad [c,d] = -\hbar c.$$

The nonzero numerical parameter $\hbar$ is introduced into the Lie brackets for future convenience. This parameter can evidently be equated to one by the renormalization of the generators. In this case we come to the conventional Lie algebra $gl(2)$.

Let us form a 2×2 matrix L whose entries are the above generators

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

[7] Let us note that K_0 of the NC sphere equipped only with the additive structure was calculated in [H]. We are rather interested in quantum line bundles. Once they are defined other projective modules can be introduced as their direct sums.

It is a matter of straightforward checking that this matrix satisfies the following second-order polynomial identity:

$$L^2 - (\mathrm{tr} + \hbar)L + (\triangle + \hbar\, \mathrm{tr}/2)\mathrm{id} = 0 \tag{2.1}$$

where

$$\mathrm{tr} = \mathrm{tr}\, L = a + d \qquad \triangle = ad - (bc + cb)/2.$$

As $\hbar \to 0$ we get the classical CH identity for a matrix with commutative entries.

In order to avoid any confusion we want to stress that we only deal with the enveloping algebra $U(gl(2)_\hbar)$ (and some of its quotients) and we disregard its (restricted) dual object—the algebra of functions on the Lie group $GL(2)$. Our immediate aim consists in constructing some 'derived' matrices with entries from $U(gl(2)_\hbar)$ satisfying some 'higher' CH identities. It will be done by some sort of coproduct applied to the matrix L and restricted onto the symmetric component. However, we do not use any coalgebraic (and hence any Hopf) structure of the algebra $U(gl(2)_\hbar)$ itself.

Remark 1. A version of the CH identity for matrices with entries from $U(gl(n))$ has been known for a long time (cf [Go]). However, traditionally one deals with the CH identity in a concrete representation of the algebra $gl(n)$ while we prefer to work with the above universal form of the CH identity. A way of obtaining the CH identity by means of the so-called Yangians was suggested in [NT]. In [G-T] another NC version of the CH identity was presented. The coefficients of the polynomial relation suggested that there are diagonal (not scalar) matrices. But such a form of the CH identity is not suitable for our aims.

Since the elements tr and $\triangle$ from (2.1) belong to the centre $Z[U(gl(2)_\hbar)]$ of the algebra $U(gl(2)_\hbar)$ one can consider the quotient

$$\mathcal{A}_\hbar = U(gl(2)_\hbar)/\{\mathrm{tr}, \triangle - \alpha\} \qquad \alpha \in \mathbb{K}.$$

Taking into consideration the fact that the trace of the matrix L vanishes we can also treat this algebra as quotient (1.5). In what follows the algebra $\mathcal{A}_\hbar$ will be called a NC variety (or more precisely, a NC hyperboloid).

Being restricted to the algebra $\mathcal{A}_\hbar$ the CH identity becomes a polynomial relation in L with numerical coefficients

$$L^2 - \hbar L + \alpha\, \mathrm{id} = 0. \tag{2.2}$$

Denote λ_1 and λ_2 as the roots of corresponding polynomial equation (see (1.4)) that is

$$\lambda_1 = (\hbar - \sqrt{\hbar^2 - 4\alpha})/2 \qquad \lambda_2 = (\hbar + \sqrt{\hbar^2 - 4\alpha})/2.$$

Let us suppose that $\lambda_1 \neq \lambda_2$. If $\hbar = 0$ this condition means that the cone corresponding to the case $\triangle = \alpha = 0$ is forbidden. However, if $\hbar \neq 0$ we have $\hbar^2 - 4\alpha \neq 0$.

We say that λ_1 and λ_2 are eigenvalues of the matrix L. Of course, this does not mean the existence of an invertible matrix $A \in M_2(\mathcal{A}_\hbar)$ such that the matrix $A \cdot L \cdot A^{-1}$ becomes diagonal: $\mathrm{diag}(\lambda_1, \lambda_2)$.

Remark 2. If an algebra $\mathcal{A}$ is not a field, the invertibility of a matrix $A \in M_n(\mathcal{A})$ is an exceptional situation. As follows from the CH identity for the matrix L it is invertible. However, it is not so even for small deformations of L.

It is easy to see that the matrices

$$e_{10} = (\lambda_2\, \mathrm{id} - L)/(\lambda_2 - \lambda_1) \qquad e_{01} = (\lambda_1\, \mathrm{id} - L)/(\lambda_1 - \lambda_2) \in M_2(\mathcal{A}_\hbar) \tag{2.3}$$

are idempotents and $e_{10} \cdot e_{01} = 0$.

Denote $E_1(\hbar)$ and $E_2(\hbar)$ as the projective modules (also called q.l.b.) corresponding, respectively, to the idempotents e_{10} and e_{01} in (2.3). Let us explicitly describe these modules.

Let $V_{(k)}$ be the kth homogeneous component $\mathrm{Sym}^k(V)$ of the symmetric algebra $\mathrm{Sym}(V)$ of the space V. Thus, $k = 2\times$ spin and $\dim(V_{(k)}) = k + 1$. Consider the tensor product $V \otimes \mathcal{A}_\hbar$. It is nothing but the free right $\mathcal{A}_\hbar$-module $\mathcal{A}_\hbar{}^{\oplus 2}$. We can imagine the matrix L (as well as any polynomial in it) as an operator acting from V to $V \otimes \mathcal{A}_\hbar$ which can be presented in a basis (v_1, v_2) as follows:

$$(v_1, v_2) \mapsto (v_1 \otimes a + v_2 \otimes c, v_1 \otimes b + v_2 \otimes d).$$

Following [GS] we define the projective module $E_l(\hbar), l = 1, 2$ as a quotient of $V \otimes \mathcal{A}_\hbar$ over its submodule generated by the elements

$$v_1 \otimes a + v_2 \otimes c - v_1 \otimes \lambda_l \qquad v_1 \otimes b + v_2 \otimes d - v_2 \otimes \lambda_l. \tag{2.4}$$

Also, the module $E_1(\hbar)$ (resp. $E_2(\hbar)$) can be identified with the image of the idempotent e_{10} (resp. e_{01}) which consists of the elements

$$(v_1 \otimes a + v_2 \otimes c - v_1 \otimes \lambda_m) f_1 + (v_1 \otimes b + v_2 \otimes d - v_2 \otimes \lambda_m) f_2 \qquad \forall f_1, f_2 \in \mathcal{A}_\hbar$$

where $m = 2$ for $E_1(\hbar)$ and $m = 1$ for $E_2(\hbar)$. (In a similar way we can associate left projective modules with the idempotent in question.)

Now, consider the compact form of the NC variety in question. Changing the basis

$$x = \mathrm{i}(d - a)/2 = -\mathrm{i}a \qquad y = \mathrm{i}(b + c)/2 \qquad z = (b - c)/2$$

we get the following commutation relations between the new generators:

$$[x, y] = \hbar z \qquad [y, z] = \hbar x \qquad [z, x] = \hbar y \tag{2.5}$$

and the defining equation of the NC variety reads now

$$\triangle = (x^2 + y^2 + z^2) = \alpha.$$

Thus, assuming $\mathbb{K} = \mathbb{R}$ and $\alpha > 0$ we get a NC analogue of the sphere, namely, the algebra

$$\mathcal{A}_\hbar = U(su(2)_\hbar)/\{x^2 + y^2 + z^2 - \alpha\}.$$

However, the eigenvalues of equation (2.2) are imaginary (for positive α and real and small enough $\hbar$) and as usual we should consider the idempotents and related projective modules over the field $\mathbb{C}$.

Completing this section we want to stress that equations (2.4) are covariant w.r.t. the action of the group G where $G = SU(2)$ or $G = SL(2)$ depending on the form (compact or not) we are dealing with.

3. Derived quantum line bundles

In this section we discuss the problem of extension of the matrix $L = L_{(1)}$ to the higher spins and suggest a method of finding the corresponding CH identities.

First, consider the commutative case. Let

$$\Delta(L) = L \otimes \mathrm{id} + \mathrm{id} \otimes L \in M_4(\mathcal{A}_\hbar) \tag{3.1}$$

be the first extension of the matrix L to the space $V^{\otimes 2}$. If λ_1 and λ_2 are (distinct) eigenvalues of L and $u_1, u_2 \in V$ are corresponding eigenvectors, then the spectrum of $\Delta(L)$ is

$$2\lambda_1 \quad (u_1 \otimes u_1) \qquad 2\lambda_2 \quad (u_2 \otimes u_2) \qquad \lambda_1 + \lambda_2 \quad (u_1 \otimes u_2 \text{ and } u_2 \otimes u_1)$$

where in brackets we indicate the corresponding eigenvectors.

The commutativity of entries of the matrix L can be expressed by the relation

$$L_1 \cdot L_2 = L_2 \cdot L_1 \qquad \text{where} \quad L_1 = L \otimes \mathrm{id} \quad L_2 = \mathrm{id} \otimes L.$$

Rewriting (3.1) in the form $\Delta(L) = L_1 + L_2$ and taking into account the CH identity for L

$$0 = (L - \lambda_1\,\mathrm{id})(L - \lambda_2\,\mathrm{id}) = L^2 - \mu L + \nu\,\mathrm{id} \qquad \mu = \lambda_1 + \lambda_2 \qquad \nu = \lambda_1\lambda_2$$

we find

$$\Delta(L)^2 = \mu\Delta(L) + 2L_1 \cdot L_2 - 2\nu\,\mathrm{id}$$
$$\Delta(L)^3 = (\mu^2 - 4\nu)\Delta(L) + 6\mu L_1 \cdot L_2 - 2\mu\nu\,\mathrm{id}.$$

Note that $\mu = 0$ if $L \in sl(2)$. Upon excluding $L_1 \cdot L_2$ from the above equations we get

$$\Delta(L)^3 - 3\mu\Delta(L)^2 + (2\mu^2 + 4\nu)\Delta(L) - 4\mu\nu\,\mathrm{id} = 0.$$

Substituting the values of μ and ν we can present this relation as follows:

$$(\Delta(L) - 2\lambda_1)(\Delta(L) - \lambda_1 - \lambda_2)(\Delta(L) - 2\lambda_2) = 0.$$

Thus, the minimal polynomial for $\Delta(L)$ is of degree 3.

A similar statement is valid for further extensions of the matrix L:

$$\Delta^2(L) = L \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes L \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes L$$

and so on. Namely, the matrix $\Delta^k(L)$ satisfies the CH identity whose roots are $k_1\lambda_1 + k_2\lambda_2$ with $k_1 + k_2 = k + 1$, and the multiplicity of each root is $C_{k+1}^{k_1}$. To avoid this multiplicity it suffices to consider *the symmetric component* (denoted as $L_{(k)}$) of the matrix $\Delta^{k+1}(L)$ (see below). Finally, the matrix $L_{(k)}$ has $k + 1$ pairwise distinct eigenvalues and its characteristic polynomial equals to that of $\Delta^k(L)$. Then by the same method as above we can associate with this matrix $k + 1$ idempotents and corresponding projective modules. Thus, we have realized the line bundles E^{k_1,k_2} under the guise of projective modules.

Now, let us pass to the NC variety in question. With matrices L_1 and L_2 the commutation relation (1.2) takes the form

$$L_1 \cdot L_2 - L_2 \cdot L_1 = \hbar(L_1 P - P L_1) \tag{3.2}$$

where P is the usual flip. So, we cannot apply the commutative binomial formula for calculating the powers of $\Delta^k(L)$. This prevents us from calculating the CH identities for the matrices $\Delta^k(L)$ with the above method.

Instead, we will calculate the CH identities directly for *the symmetric components* of these matrices, also denoted $L_{(k)}$, $k = 2, 3, \ldots$ and defined as

$$L_{(k)} = kS^{(k)} L_1 S^{(k)}$$

where $S^{(k)}$ is the Young symmetrizer in $V^{\otimes k}$. Taking into consideration that the element $S^{(k)}L_1$ is already symmetrized w.r.t. all factors apart from the first one we can represent the matrix $L_{(k)}$ as

$$L_{(k)} = S^{(k)} L_1(\mathrm{id} + P^{12} + P^{12}P^{23} + \cdots + P^{12}P^{23}\cdots P^{k-1k})$$

where P^{ii+1} is the operator transposing the ith and $(i + 1)$th factors in the tensor product of spaces.

Remark 3. We can treat the matrix $L_{(k)}$ as an operator acting from $V^{\otimes k}$ to $V^{\otimes k} \otimes \mathcal{A}_\hbar$ assuming it to be trivial on all components except for $V_{(k)} \subset V^{\otimes k}$.

In case $k = 2$ we have the following proposition.

Proposition 4. *The CH identity for the matrix $L_{(2)}$ restricted to the symmetric component $V_{(2)}$ of the space $V^{\otimes 2}$ is*

$$L_{(2)}^3 - 4\hbar L_{(2)}^2 + 4(\alpha + \hbar^2)L_{(2)} - 8\hbar\alpha\,\mathrm{id} = 0. \qquad (3.3)$$

Proof. By definition the matrix $L_{(2)}$ has the form

$$L_{(2)} = \tfrac{1}{2}(\mathrm{id} + P^{12})L_1(\mathrm{id} + P^{12}).$$

Taking into consideration that $L_2 = P^{12}L_1P^{12}$ we rewrite (3.2) as follows:

$$L_1P^{12}L_1P^{12} - P^{12}L_1P^{12}L_1 = \hbar(L_1P^{12} - P^{12}L_1). \qquad (3.4)$$

Now we need some powers of the matrix $L_{(2)}$. We find ($L \equiv L_1$, $P \equiv P^{12}$):

$$L_{(2)}^2 = \tfrac{1}{2}(\mathrm{id} + P)L(\mathrm{id} + P)L(\mathrm{id} + P)$$
$$L_{(2)}^3 = \tfrac{1}{2}(\mathrm{id} + P)L(\mathrm{id} + P)L(\mathrm{id} + P)L(\mathrm{id} + P).$$

Then taking into account (2.2) and (3.4) we have the following chain of identical transformations for $L_{(2)}^3$:

$$\begin{aligned}
2L_{(2)}^3 &= (\mathrm{id} + P)L\underline{(\mathrm{id} + P)L(\mathrm{id} + P)L}(\mathrm{id} + P)\\
&= (\mathrm{id} + P)L[\underline{PLPL} + PL^2 + LPL + L^2](\mathrm{id} + P)\\
&= 2(\mathrm{id} + P)L[LPL + \hbar PL - \alpha\,\mathrm{id}](\mathrm{id} + P)\\
&= 4\hbar(\mathrm{id} + P)LPL(\mathrm{id} + P) - 4\alpha(\mathrm{id} + P)L(\mathrm{id} + P)\\
&= 4\hbar(\mathrm{id} + P)L(\mathrm{id} + P)L(\mathrm{id} + P) - 4\hbar(\mathrm{id} + P)\\
&\quad\times(\hbar L - \alpha\,\mathrm{id})(\mathrm{id} + P) - 4\alpha(\mathrm{id} + P)L(\mathrm{id} + P)\\
&= 8\hbar L_{(2)}^2 - 8(\hbar^2 + \alpha)L_{(2)} + 8\hbar\alpha(\mathrm{id} + P).
\end{aligned}$$

Cancelling the factor 2 we come to the result

$$L_{(2)}^3 - 4\hbar L_{(2)}^2 + 4(\alpha + \hbar^2)L_{(2)} - 4\hbar\alpha(\mathrm{id} + P) = 0. \qquad (3.5)$$

To complete the proof it remains to note, that after restriction to the symmetric component $V_{(2)}$ of the space $V^{\otimes 2}$ the last term in (3.5) turns into $8\hbar\alpha$ id and we come to (3.3). □

Now, let us exhibit the matrix $L_{(2)}$ in a basis form. In the base

$$v_{20} = v_1^{\otimes 2} \qquad v_{11} = v_1 \otimes v_2 + v_2 \otimes v_1 \qquad v_{02} = v_2^{\otimes 2}$$

of the space $V_{(2)}$ this matrix has the following form:

$$L_{(2)} = \begin{pmatrix} 2a & 2b & 0\\ c & a+d & b\\ 0 & 2c & 2d \end{pmatrix} = \begin{pmatrix} 2a & 2b & 0\\ c & 0 & b\\ 0 & 2c & -2a \end{pmatrix} \qquad (3.6)$$

(the latter equality holds in virtue of the condition $a + d = 0$).

In the following we prefer to deal with another basis in the space $V_{(2)}$. Namely, by putting

$$(v_{20}, v_{11}, v_{02}) = (u_{20}, u_{11}, u_{02})\mathbf{P}$$

with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0\\ 1/2 & 0 & -1/2\\ -\mathrm{i}/2 & 0 & -\mathrm{i}/2 \end{pmatrix}$$

we transform $L_{(2)}$ into the form

$$\overline{L}_{(2)} = \mathbf{P}L_{(2)}\mathbf{P}^{-1} = 2\begin{pmatrix} 0 & -z & y\\ z & 0 & -x\\ -y & x & 0 \end{pmatrix}. \qquad (3.7)$$

The matrix $\overline{L}_{(2)}$ is expressed through the generators (x, y, z) and it is better adapted to the compact form of the NC varieties in question. However, it satisfies the same NC version of CH identity (2.2).

Remark 5. By straightforward checking we can see that the roots of (3.3) are

$$\lambda_{20} = \hbar - \sqrt{\hbar^2 - 4\alpha} = 2\lambda_1\lambda_{11} = 2\hbar = 2(\lambda_1 + \lambda_2) \qquad \text{and} \quad \lambda_{02} = \hbar + \sqrt{\hbar^2 - 4\alpha} = 2\lambda_2. \tag{3.8}$$

These quantities are eigenvalues of the matrix (3.6) which is the restriction of the matrix (3.1) to the symmetric part of the space $V^{\otimes 2}$. A similar restriction of the matrix (3.1) to the skewsymmetric part of the space $V^{\otimes 2}$ gives rise to the operator

$$v_1 \otimes v_2 - v_2 \otimes v_1 \to (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes (a + d) = 0.$$

Thus, the matrix (3.1) on the whole space $V^{\otimes 2}$ has four distinct eigenvalues (those of (3.8) and 0) and in contrast with the commutative case its minimal polynomial cannot be of the third degree.

As for higher extensions $L_{(k)}$, $k > 2$, of the matrix L the following holds.

Proposition 6. *For any integer $k > 2$ there exists a polynomial*

$$P_k(x) = \lambda^k + \sum_{i=1}^{k} a_{k-i}\lambda^{k-i}$$

with numerical coefficients such that $P_k(L_{(k)}) = 0$. Moreover, its roots are

$$\lambda_{k_1k_2} = k_1\lambda_1 + k_1k_2(\lambda_1 + \lambda_2) + k_2\lambda_2 \qquad k_1 + k_2 = k.$$

This formula can be deduced from [Ro]. We will present its q-analogue in [GLS].

Similar to the basic case the roots of the polynomial P_k will be called eigenvalues of the corresponding matrix $L_{(k)}$.

Now, let us assume the eigenvalues $\lambda_{20}, \lambda_{11}, \lambda_{02}$ to be also pairwise distinct. Then, by using the same method as above we can introduce the idempotent

$$e_{20} = (\lambda_{11}\,\mathrm{id} - L_{(2)})(\lambda_{02}\,\mathrm{id} - L_{(2)})/(\lambda_{11} - \lambda_{20})(\lambda_{02} - \lambda_{20})$$

and similarly e_{11} and e_{02} corresponding to the eigenvalues λ_{11}, and λ_{02}, respectively. The related q.l.b. (projective $\mathcal{A}_\hbar$-modules) will be denoted $E^{20}(\hbar)$, $E^{11}(\hbar)$ and $E^{02}(\hbar)$, respectively.

Assuming the eigenvalues of the polynomials P_k, $k > 2$ (see proposition 6) to be also pairwise distinct we can associate with the matrix $L_{(k)}$ $k + 1$ idempotents

$$e_{k_1k_2} \qquad k_1 + k_2 = k \qquad k > 0$$

and the corresponding q.l.b. $E^{k_1,k_2}(\hbar)$. For $k = 0$ we set $e_{00} = 1$. The corresponding q.l.b. is $E^{0,0}(\hbar)$.

Remark 7. Let us note that if we do not fix any value of $\triangle$ we can treat the elements e_{ij} as those of $M_2(U(sl(2)_\hbar)) \otimes R$ where R is the field of fractions of the algebraic closure $\overline{Z[U(sl(2)_\hbar)]}$.

Remark 8. Note that there exists a natural generalization of the above constructions giving rise to some 'braided varieties' and corresponding 'line bundles' as follows. Let R be a Hecke symmetry of rank 2 (cf [G]). Then the matrix L satisfying the RE with such R obeys an equation analogous to (2.1) but with appropriate trace and determinant (cf [GPS]). Introducing the quotient algebra $\mathcal{A}_\hbar$ in a similar way we treat it as a braided analogue of a NC sphere.

Then, by defining the extensions $L_{(k)}$ as above (but with a modified meaning of the symmetric powers of the space V) we can define a family of q.l.b. over such a 'NC braided variety' as above. This construction will be presented in detail in [GLS].

Now, we pass to computing the quantities $\mathrm{tr}\, e_{k_1k_2}$.

Proposition 9. *The following relation holds:*

$$\mathrm{tr}\, e_{k_1k_2} = 1 + \frac{(k_1 - k_2)\hbar}{\sqrt{\hbar^2 - 4\alpha}}.$$

A proof of this formula will be given in [GLS] in a more general context including its q-analogue.

4. Isomorphic modules and multiplicative structure

First of all we discuss the problem of isomorphism between the projective modules introduced above (namely, q.l.b.). There exists a number of definitions of isomorphic modules over $\mathbb{C}^*$-algebras (cf [W]). However, for the algebras in question we use the following definition motivated by [R].

Definition 10. *We say that two projective modules $M_1 \subset \mathcal{A}^{\oplus m}$ and $M_2 \subset \mathcal{A}^{\oplus n}$ over an algebra $\mathcal{A}$ corresponding to the idempotents e_1 and e_2, respectively are isomorphic iff there exist two matrices $A \in M_{m,n}(\mathcal{A})$ and $B \in M_{n,m}(\mathcal{A})$ such that*

$$AB = e_1 \qquad BA = e_2 \qquad A = e_1A = Ae_2 \qquad B = e_2B = Be_1.$$

Proposition 11. *The q.l.b. $E^{1,1}(\hbar)$ is isomorphic to $E^{0,0}(\hbar)$.*

Proof. It is not difficult to see that

$$e_{11} = \left(L_{(2)}^2 - 2\hbar L_{(2)} + 4\alpha\, \mathrm{id}\right)\big/(4\alpha).$$

Then, by straightforward calculations we check that for the idempotent e_{11} the following relation holds:

$$(4\alpha)^{-1}\left(\begin{pmatrix} 4a^2 + 2bc & 4ab & 2b^2 \\ 2ca & 2cb + 2bc & -2ba \\ 2c^2 & -4ac & 2cb + 4a^2 \end{pmatrix} - 2\hbar \begin{pmatrix} 2a & 2b & 0 \\ c & 0 & b \\ 0 & 2c & -2a \end{pmatrix} + 4\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$
$$= (4\alpha)^{-1} \begin{pmatrix} -2b \\ 2a \\ 2c \end{pmatrix} (c \quad -2a \quad -b)\,.$$

Passing to the matrix $\overline{L}_{(2)}$ we get

$$e_{11} = (4\alpha)^{-1}\left(\overline{L}_{(2)}^2 - 2\hbar \overline{L}_{(2)} + 4\alpha\, \mathrm{id}\right) = \alpha^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x \quad y \quad z)$$

(So, the idempotent e_{11} defines the following operator in $\mathcal{A}^{\oplus 3}$:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \mapsto (\alpha)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x \cdot f_1 + y \cdot f_2 + z \cdot f_3).)$$

It remains to say that if we put $A = (\alpha)^{-1}(x \quad y \quad z)$ and $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we satisfy the above definition with $e_1 = e_{00}$ and $e_2 = e_{11}$. □

In general, the problem of isomorphism between modules (1.6) is open. We can only conjecture that the q.l.b. (1.6) are isomorphic to each other.

Remark 12. If the algebra $\mathcal{A}$ is not commutative the first two relations of the definition 10 do not yield the equality $\mathrm{tr}\, e_1 = \mathrm{tr}\, e_2$. However, proposition 9 implies that

$$\mathrm{tr}\, e_{k_1,k_2} = \mathrm{tr}\, e_{k_1+l,k_2+l}.$$

Now, we can introduce an associative product on the set of q.l.b. in a natural way by setting

$$E^{k_1,k_2}(\hbar) \cdot E^{l_1,l_2}(\hbar) = E^{k_1+l_1,k_2+l_2}(\hbar).$$

This product is evidently associative and commutative. In particular, we have

$$E_1(\hbar) \cdot E_2(\hbar) = E^{1,1}(\hbar) \qquad E_1(\hbar) \cdot E_1(\hbar) = E^{2,0}(\hbar) \qquad E_2(\hbar) \cdot E_2(\hbar) = E^{0,2}(\hbar).$$

The family of the modules $E^{k_1,k_2}(\hbar)$ equipped with this product is denoted $\mathrm{prePic}(\mathcal{A}_\hbar)$ and called *prePicard semigroup* of the NC sphere.

Assuming that the q.l.b. (1.6) are indeed isomorphic to each other we can naturally define the Picard group $\mathrm{Pic}(\mathcal{A}_\hbar)$ of the NC sphere as the classes of isomorphic modules $E^{k_1,k_2}(\hbar)$ equipped with the above product.

So, under this assumption, the Picard group $\mathrm{Pic}(\mathcal{A}_\hbar)$ of the NC sphere is at most $\mathbb{Z}$ (recall that $\mathrm{Pic}(S^2) = \mathbb{Z}$).

Now, consider the problem of computing the pairing

$$\langle\, ,\, \rangle : \mathrm{prePic}(\mathcal{A}_\hbar) \otimes K^0 \to \mathbb{K}. \tag{4.1}$$

Such a pairing plays a key role in the Connes version of the index formula. (Usually, one considers K_0 instead of prePic but we restrict ourselves to 'quantum line bundles'. Moreover, assuming the conjecture formulated before remark 12 to be true we can replace prePic in formula (4.1) by Pic.) Let us recall that K^0 stands for the Grothendieck ring of the category of irreducible modules of the algebra in question[8]. In the spirit of the NC index (cf [L]) the pairing (4.1) can be defined as

$$\langle E^{k_1,k_2}(\hbar), U\rangle = \mathrm{tr}\, \pi_U(\mathrm{tr}(e_{k_1,k_2})) \tag{4.2}$$

where $\mathrm{tr}(e_{k_1,k_2}) \in \mathcal{A}_\hbar$ and $\pi_U : \mathcal{A}_\hbar \to \mathrm{End}(U)$ is the representation corresponding to the irreducible U.

It is not difficult to see that the result of the pairing of the module $E^{k_1,k_2}(\hbar)$ with the irreducible U_j of the spin j is equal to the quantity $n\, \mathrm{tr}\, e_{k_1,k_2}$ evaluated at the point

$$\alpha = -\hbar^2(n^2-1)/4 \qquad \text{where} \quad n = \dim U_j = 2j+1. \tag{4.3}$$

In particular, we have

$$\langle E^{0,0}(\hbar), U_j\rangle = n \qquad \langle E^{1,0}(\hbar), U_j\rangle = n+1 \qquad \langle E^{0,1}(\hbar), U_j\rangle = n-1$$
$$\langle E^{2,0}(\hbar), U_j\rangle = n+2 \qquad \langle E^{0,2}(\hbar), U_j\rangle = n-2$$

(here we assume that $\sqrt{n^2\hbar^2} = n\hbar$). More generally, if $n > k_1 + k_2$ we have

$$\langle E^{k_1,k_2}(\hbar), U_j\rangle = k_1 - k_2 + n.$$

This formula follows immediately from proposition 9. There exists a q-analogue of this formula which will be considered in [GLS].

[8] By this we mean the Grothendieck ring of the algebra $U(su(2)_\hbar)$ or what is the same $U(su(2))$ since $\hbar$ does not matter here. However, any irreducible $U(su(2)_\hbar)$-module defines a relation between $\hbar$ and α (see (4.3)) and, therefore, the factors in formula (4.2) are not independent.

5. Projective modules in differential calculus

In this section we define NC analogues of some vector bundles arising in differential calculus on a classical sphere such as tangent vector bundle and those of differential forms. Also, we suggest a new way of defining the NC analogue of the de Rham complex.

Let us begin with a NC analogue of the tangent bundle $T(S^2)$ on the sphere. Since this bundle is complementary to the normal one and since the latter bundle (treated as a module) is nothing but $E^{1,1}$, it is natural to define the NC analogue of $T(S^2)$ as

$$T(\mathcal{A}_\hbar) = E^{2,0}(\hbar) + E^{0,2}(\hbar).$$

We call it *a tangent module* on the NC sphere.

This module can be represented by the equation $\mathrm{Im}\, e_{11} = 0$. It is equivalent to the relation

$$u_{20}x + u_{11}y + u_{02}z = 0. \tag{5.1}$$

This means that the module $T(\mathcal{A}_\hbar)$ is realized as the quotient of the free module $\mathcal{A}_\hbar^{\oplus 3}$ generated by the elements

$$u_{20} \quad u_{11} \quad u_{02} \tag{5.2}$$

over the submodule $\{Cf,\ f \in \mathcal{A}_\hbar\}$ where C is the lhs of (5.1).

In the classical case relation (5.1) is motivated by an operator meaning of the tangent space. Namely, if generators (5.2) are treated as infinitesimal rotations of the sphere[9] and the symbols x, y, z in (5.1) are considered as operators of multiplication on the corresponding functions, then the element C treated as an operator is trivial. This allows us to equip the tangent module $T(S^2)$ with an operator meaning by converting any element of this module into a vector field. Thus, we get the action

$$\mathcal{A} \otimes T(S^2) \to \mathcal{A} \qquad \mathcal{A} = \mathbb{K}(S^2)$$

which consists in applying a vector field to a function.

However, if we assign the same meaning to generators (5.2) and to x, y, z on the NC sphere (by setting $u_{20} = [x, \cdot]$ and so on and considering x, y, z as operators of multiplication on the corresponding generator) then the element C treated as an operator is no longer trivial. This is the reason why we are not able to provide the tangent module $T(\mathcal{A}_\hbar)$ with a similar action on the algebra $\mathcal{A}_\hbar$.

Remark 13. We want to stress that the space of derivations of the algebra $\mathcal{A}_\hbar$ often considered as a proper NC counterpart of the usual tangent space does not have any $\mathcal{A}_\hbar$-module structure. So, for a NC variety which is a deformation of a classical one we have a choice: which properties of the classical object we want to preserve. In our approach we prefer to keep the property of the tangent space to be a projective module. In the same manner we will treat the terms of the NC de Rham complex (see below).

Passing to the cotangent bundle $T^*(S^2)$ or otherwise stated to the space of the first-order differentials $\Omega^1(S^2)$ we see that it is isomorphic to $T(S^2)$. Therefore, it is defined by the same relation (5.1) but with another meaning of generators (5.2): now we treat them as the differentials of the functions x, y and z, respectively:

$$u_{20} = \mathrm{d}x \qquad u_{11} = \mathrm{d}y \qquad u_{02} = \mathrm{d}z.$$

This gives us the motivation to introduce a *cotangent module* $T^*(\mathcal{A}_\hbar)$ on the NC sphere similar to the tangent one but with a new meaning of generators (5.2). We will also use the notation $\Omega^1(\mathcal{A}_\hbar)$ for the module $T^*(\mathcal{A}_\hbar)$.

[9] By means of the Kirillov bracket we can represent these rotations as $u_{20} = \{x, \cdot\}$ and so on.

Up to now we have no modifications in the relations defining the tangent and cotangent objects. Nevertheless, it is no longer so for the NC counterpart of the second-order differential space $\Omega^2(S^2)$. In the classical case this space is defined by

$$u_{11}z - u_{02}y = 0 \qquad -u_{20}z + u_{02}x = 0 \qquad u_{20}y - u_{11}x = 0 \tag{5.3}$$

with the following meaning of generators (5.2)

$$u_{20} = \mathrm{d}y \wedge \mathrm{d}z \qquad u_{11} = \mathrm{d}z \wedge \mathrm{d}x \qquad u_{02} = \mathrm{d}x \wedge \mathrm{d}y. \tag{5.4}$$

In a concise form we can rewrite (5.3) as

$$(u_{20}, u_{11}, u_{02}) \cdot \overline{L}_{(2)} = 0 \tag{5.5}$$

with $\overline{L}_{(2)}$ given by (3.7).

However, in the NC case we should replace (5.5) by

$$(u_{20}, u_{11}u_{02}) \cdot \overline{L}_{(2)} = 2\hbar(u_{20}, u_{11}u_{02})$$

or in a more detailed form

$$u_{11}z - u_{02}y = \hbar u_{20} \qquad -u_{20}z + u_{02}x = \hbar u_{11} \qquad u_{20}y - u_{11}x = \hbar u_{02}.$$

This is motivated by the fact that $2\hbar$ becomes an eigenvalue of the matrix $\overline{L}_{(2)}$. Let us denote the corresponding quotient module $\Omega^2(\mathcal{A}_\hbar)$. Of course, we can represent generators (5.2) in the form (5.4) but now it does not have any sense.

We emphasize that we do not treat elements of the $\mathcal{A}_\hbar$-modules $\Omega^i(\mathcal{A}_\hbar), i = 1, 2$ as differential forms. By contrast, we preserve the basic property of the $\mathcal{A}$-modules $\Omega^i(\mathcal{A}), i = 1, 2$ to be projective.

This allows us to conclude that the $\mathcal{A}_\hbar$-modules $\Omega^i(\mathcal{A}_\hbar), i = 1, 2$ contain (at least for a generic $\hbar$) just the same irreducible $SU(2)$ components as $\mathcal{A}$-modules $\Omega^i(\mathcal{A}), i = 1, 2$. Explicitly these components are listed in [AG].

By using this property we can construct $SU(2)$-covariant isomorphisms of $\mathbb{K}[[\hbar]]$-modules

$$\psi_i(\hbar) : \Omega^i(\mathcal{A}_\hbar) \to \Omega^i(\mathcal{A}) \otimes \mathbb{K}[[\hbar]]$$

where the tensor product is completed in $\hbar$-adic topology (we can also impose the property $\psi_i(\hbar) = \mathrm{id} \bmod \hbar$). It is well known that a similar isomorphism $\psi_0(\hbar) : \mathcal{A}_\hbar \to \mathcal{A} \otimes \mathbb{K}[[\hbar]]$ exists due to the deformation nature of the algebra $\mathcal{A}_\hbar$.

Then we introduce a NC analogue

$$0 \to \Omega^0(\mathcal{A}_\hbar) = \mathcal{A}_\hbar \to \Omega^1(\mathcal{A}_\hbar) \to \Omega^2(\mathcal{A}_\hbar) \to 0 \tag{5.6}$$

of the classical de Rham complex by drawing the differential d from the classical complex (tensorized by $\mathbb{K}[[\hbar]]$) via the above isomorphisms:

$$\mathrm{d}_{NC} = \psi(\hbar)^{-1}\, \mathrm{d}\psi(\hbar) \qquad \psi(\hbar) = \{\psi_i(\hbar)\}.$$

Note that the property $\mathrm{d}_{NC}^2 = 0$ is fulfilled automatically whereas the Leibniz rule fails (as well as the multiplicative structure of the space of all differential forms[10]).

We want to complete the paper with the following short survey of the existing approaches to differential calculus on NC algebras.

Remark 14. Actually there are two different generalizations known of differential calculus associated with NC algebras. The first one makes use of the so-called universal differential algebra. This algebra is generated by elements of the initial algebra $\mathcal{A}$ and formal differentials

[10] In the classical case the multiplicative structure of the space $\Omega(\mathcal{A}) = \bigoplus \Omega^i(\mathcal{A})$ is ensured by the fact that all one-sided modules over a commutative algebra are automatically two-sided. However, this is not the case for the algebras in question.

da, $a \in \mathcal{A}$ without any commutation rule between them looking like the classical one. The only Leibniz rule is used in order to transpose a differential from left to right of a 'function' and vice versa. However, the universal differential algebra is much bigger than the classical one even if the algebra $\mathcal{A}$ is commutative such as the coordinate ring of a given regular variety (cf for example [GVF]). Moreover, the components of such a differential algebra are not finitely generated modules. Finally, the associated de Rham complex differs drastically from the classical one.

If an algebra $\mathcal{A}$ is equipped with a Yang–Baxter operator compatible in some sense with the product, one usually tries to reduce the universal differential algebra to the 'classical size' (assuming it to be a deformation of a commutative algebra). In order to do this one looks for a transposition rule between 'functions' and differentials arising from the Yang–Baxter operator (once vector fields are defined their transposing rule with 'functions' and differentials is also of interest). If such an operator R is involutary ($R^2 = \mathrm{id}$) it is easy to do. However, if it is not so, keeping the classical size of differential algebra is not in general compatible with the Leibniz rule. We refer the reader to the paper [FP] where this problem is discussed w.r.t. the algebra of functions on a quantum group and to the paper [AG] where a quantum sphere is considered.

Note that the algebra $\mathcal{A}_\hbar$ is not equipped with any Yang–Baxter operator different from the usual flip and the second method is useless for reducing the corresponding universal differential algebra. Above we have suggested one more method of constructing a NC analogue of the de Rham complex associated with the algebra $\mathcal{A}_\hbar$ which makes use of some projective modules. This method allows us to preserve the 'classical size' of the terms of the de Rham complex. Moreover, its cohomology coincides with the classical one by construction. However, this is done at the expense of the Leibniz rule and the multiplicative structure of the space $\Omega(\mathcal{A}_\hbar) = \bigoplus \Omega^i(\mathcal{A}_\hbar)$. As we have said above, one always has to choose which properties of a classical object are to be preserved.

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